ON THE NON-EXISTENCE OF SECOND AND HIGHER ORDER NECESSARY BEST ESTIMATORS

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1. Introduction

Let U denote a population of N distinguishable units u_i associated with real numbers termed as value X_i ($i=1, 2, \ldots N$). The object is to estimate the population total

$$T_x = \sum_{i=1}^{N} x_i \qquad \ldots (1)$$

by drawing a random sample of size n from the population. The procedure of drawing a sample is varying probability without replacement. There are $\binom{N}{n}$ samples if the order of the units in the sample is not considered, if the order is taken into account then there are $\binom{N}{n} \cdot \lfloor n$ samples. The sampling design is then given by

$$D=(S, P) \qquad \ldots (1.2)$$

where S is the set of all possible samples and P stands for the corresponding set of the probabilities of the samples belonging to S.

For the sampling design (1.2) Horvitz and Thompson[3] introduced three classes of linear estimators for the population parameters and pointed out that these classes are not exhaustive classes of possible linear estimators. Koop[4] and Prabhu Ajgaonkar and Tikkiwal[5] independently showed that in all there are seven classes of linear estimators. Godambe[1] and Prabhu Ajgaonkar[8] proved that in the general class* the Minimum Variance Linear Unbiased Estimator (MVLUE) does not exist. For any class of linear estimators, when the MVLUE does not exist the less restrictive and the serviceable criterion of necessary best estimator was suggested by Prabhu Ajgaonkar[7]. He[8] defines a necessary best estimator as follows :

*The general class is synonymous of T_7-class of linear estimators.

Let t' be a class of linear unbiased estimators when the sample design D in (2) is employed. It is easily noted that the variance of t' can be expressed in the following quadratic form:

$$\text{Var}\,(t') = \sum_{i=1}^{N} A_{ii}\, Y_i^2 + \sum_{i \neq j=1}^{N} A_{ij}\, Y_i\, Y_j \qquad \ldots (a)$$

where the quantity A_{ij} $(i, j=1, 2, \ldots N)$ involve the known functions of probablities and coefficients of the class.

Further let t_1' be an unbiased estimator belonging to the class t' and that the variance of t_1' be given by

$$\text{Var}\,(t'_1) = \sum_{i=1}^{N} b_{ii}\, Y_i^2 + \sum_{i \neq j=1}^{N} b_{ij}\, Y_i\, Y_j \qquad \ldots (b)$$

consider the quantity

$$Q = \text{Var}\,(t') - \text{Var}\,(t'_1)$$

$$= \sum_{i=1}^{N} (A_{ii} - b_{ii})\, Y_i^2 + \sum_{i \neq j=1}^{N} (A_{ij} - b_{ij})\, Y_i\, Y_j$$

obtained from equations (a) and (b).

Definition :—The estimator t'_1 is a necessary best estimator of order r for the class t' if all the leading principal minors of Q up to the order r, are positive.

Hege[2] proved that the Horvitz-Thompson's[3] estimator is necessary best estimator of order one in all the classes of linear unbiased estimators.

2. Non-Existence of Necessary Best Estimator of Order Two in the T_3-Class of Linear Unbiased Estimators

In this class the weight is associated to the sample. Thus in general T_3-class estimator for the population total T_x is defined as

$$\hat{T}_3 = \gamma_{se} \sum_{r=1}^{n} (X_r = x_{i_r}) \qquad \ldots (2.1)$$

when X_r represents the outcome at the rth draw and γ_{se} is the weight associated to the se^{th} sample. Prabhu Ajgaonkar[6] while considering the non-existence of MVLUE in this class when sampling is carried out WOR obtains the following conditions of unbiasedness of $\hat{T}_3$ for population T_x

$$\sum_{r=1}^{n} \sum' w_{i_1 i_2, \ldots, i_{r-1}, j, i_{r+1}, \ldots, i_n} \cdot p_{i_1 i_2, \ldots, i_{r-1}, j, i_{r+1} \ldots, i_n} = 1$$

$$se=(i_1, i_2, \ldots i_{r-1}, j, i_{r+1}, \ldots i_n) \qquad \ldots (2.2)$$

where $\sum'$ stands for summation overall possible samples $(i_1, i_2, \ldots, i_{r-1}, j, i_{r+1}, i_n)$ in which jth unit occurs at the rth draw. Such possible samples are $(n-1)! \binom{N-1}{n-1}$ in number. Further Prabhu Ajgaonkar[6], minimising the variance of $\hat{T}_3$ subject to the condition (2.2) obtains the optimum weights as

$$w_{i_1 i_2, \ldots i_{r-1}, i_r, \ldots i_n} = \frac{\lambda_{i_1} + \lambda_{i_2} + \ldots + \lambda_{i_n}}{(x_{i_1} + x_{i_2} + \ldots + x_{i_n})^2} \qquad \ldots (2.3)$$

where λ's are the Lagrangian undetermined multipliers.

Now if we consider the population vector $X=[x_i=0$ and $x_j=0$ for $j \neq i=1, 2, \ldots N]$ and assume that ith unit occurs at the rth draw, then (2.3) gives

$$w_{j_1, j_2, \ldots j_{r-1} i_r, j_{r+1}, \ldots j_n} = \frac{\lambda_{j_1} + \lambda_{j_2} + \ldots + \lambda_{j_{r-1}} + \lambda_{i_r} + \lambda_{j_{r+1}} + . \lambda_{j_n}}{(x_{i_r})^2} \qquad \ldots (2.4)$$

Multiplying (2.4) by $p_{j_1 j_2, \ldots i_r, j_{r+1}, \ldots j_n}$ and then summing over those s_e's which contain $(j_1, j_2, j_{r-1}, i_r, j_{r+1}, \ldots j_n)$ units then we get,

$$(\lambda j_1 + \lambda j_2 + \ldots + \lambda j_{r-1} + \lambda i_r + \lambda j_{r+1} \ldots + \lambda j_n) = \frac{(x_{i_r})^2}{\pi_i} \qquad \ldots (2.5)$$

Again (2.5) and (2.4) give

$$w j_1, j_2, \ldots j_{r-1}, i_r, j_{r+1}, \ldots j_n = \frac{1}{\pi_i} \qquad \ldots (2.6)$$

Thus (2.6) suggests that the necessary best estimator of order one exists in T_3-class of linear unbiased estimators.

Again let us consider the population vector $X=(x_i \neq 0, x_j \neq 0$ and $x'_j=0 ; j' \neq i, j=1, 2, \ldots N)$. Let us now consider the four sets of w's as

$$w_{i_1, j_2', j_3', \ldots j_n'} = \frac{(\lambda_{i_1} + \lambda_{j'_2} + \ldots + \lambda_{j'_n})}{(x_{i_1} + x'_{j_2} + \ldots + x_{j'_n})^2} \qquad \ldots (2.7)$$

$$w_{j_1}, i_2, j'_3, \ldots j'_n = \frac{(\lambda_{j_1} + \lambda_{i_2} + \lambda_{j'_3} + \ldots + \lambda_{j_n}')}{(x_{j_1} + \lambda_{i_2}' + x_{j_3}' + \ldots + x_{j_n}')^2} \qquad \ldots (2.8)$$

$$w_{i_1}, j_2, j'_3, \ldots j'_n = \frac{\lambda_{i_1} + \lambda_{j_2} + \lambda_{j_3}' + \ldots + \lambda_{j_n}'}{(x_{i_1} + x_{j_2} + x_{j_3}' + \ldots + x_{j_n}')^2} \qquad \ldots (2.9)$$

$$w_{j_1}, j'_2, \ldots j'_n = \frac{\lambda_{j_1} + \lambda_{j_2}' + \ldots + \lambda_{j_n}'}{(x_{j_1} + x_{j_2}' + \ldots + x_{j_n}')^2} \qquad \ldots (2\ 10)$$

Prabhu Ajgoankar[6] has proved that the w's do not depend on the order of the units, so that equations (2.8) and (2.9) are identical. With the help of the equations (2.7), (2.8), (2.9) and (2.10) we can easily construct a quadratic, form $X'BX=0$, where B is the coefficient matrix involving w's as its elements. Thus $X'BX=0$ iff $B=0$ (a null matrix), which suggests that w's are all zero; which gives a contradiction to the conditions in equation (2.2). Thus the Necessary Best Estimator of order two in T_3-class of linear unbiased estimators does not exist.

3. Non Existence of Necessary Best Estimator of Order Two in T_4-Class of Linear Unbiased Estimators

The general estimator in T_4-class can be defined as

$$\hat{T}_4 = \sum_{r=1}^{n} \beta_{ro}\ (X_r) \qquad \ldots (3.1)$$

where $\beta_{ro} = \beta_{r\lambda}$, when $X_r = x_\lambda$ for $r=1, 2, \ldots n$; is the weight associated with the λ^{th} unit of the population when it appears at the r^{th} draw. Prabhu Ajgaonkar[9] has shown that the conditions of unbiasedness for $\hat{T}$ for the population total T_x are given by

$$\sum_{r=1}^{n} \beta_{r\lambda}\ P_{\lambda_r} = 1 \qquad \text{for} \qquad \lambda = 1, 2, \ldots N \qquad \ldots (3.2)$$

Also

$$V(\hat{T}_4) = \sum_{\lambda=1}^{N} x_\lambda^2 \sum_{r=1}^{n} \beta_{r\lambda}^2\ P_{\lambda r} + \sum_{\lambda=1}^{N} \sum_{\lambda' (\neq \lambda)=1}^{N}$$

$$\cdot\ x_\lambda x_{\lambda'} \sum_{r=1}^{n} \sum_{s(\neq r)=1}^{n} \beta_{r\lambda}\ \beta_{s\lambda'}\ p(\lambda_r, \lambda_s') - T_x^2 \ldots (3.3)$$

Minimising the R.H.S. of (3.3) w.r.t. (3.2), we get

$$\beta_{r\lambda}\, x^2_{\lambda}\, p_{\lambda r} + x_{\lambda} \sum_{s\,(\neq r)=1}^{n} \quad \sum_{\lambda'\,(\neq\lambda)=1}^{N} x_{\lambda}'\, \beta_s\lambda'\, p\,(\lambda_r, \lambda_s') = u_{\lambda} \cdot p_{\lambda r} \qquad \dots (3.4)$$

for all r and λ, where u_{λ} is the Lagrangian undetermined multiplier. With the help of equation (3.4) Prabhu Ajgaonkar[9] proves that under certain probability systems the minimum variance linear unbiased estimator independent of population values exists in this class but not otherwise and then he considers the criterion of necessary best estimator in this class and proved the existence of necessary best estimator of order one. But Tikkiwal[11] proved in general that the MVLUE independent of population values does not exist for any probability system. So the present authors have investigated the non-existence of necessary best estimator of order two in this class. To maintain the continuity, we also derive the necessary best estimator of order one.

Consider the population vector $X=(x_{\lambda}\neq 0,\ x'_{\lambda}=0$ for $\lambda'\neq\lambda=1, 2, \dots N)$. Then equation (3.4) gives

$$\beta_{r\lambda}\, x^2_{\lambda}\, p_{\lambda r} = u_{\lambda}\, p_{\lambda r} \qquad \dots (3.5)$$

Summing equation (3.5) over r, we get

$$u_{\lambda} = \frac{x_{\lambda}^{2}}{\Sigma p_{\lambda r}} = \frac{x_{\lambda}^{2}}{\pi_{\lambda}} \qquad \dots (3.6)$$

From (3.6) and (3.5) we get

$$\beta_{r\lambda} = \frac{1}{\pi_{\lambda}} \text{ for all } r \text{ and } \lambda. \qquad \dots (3.7)$$

Equation (3.7) suggests that the necessary best estimator of order one in T_4-class exists.

Now if we consider the population vector $X=(x_{\lambda}\neq 0, x_{\lambda}'\neq 0, x_{\lambda}''=0$ for $\lambda''\,(\neq\lambda, \lambda')=1, 2, \dots N)$. Then from equation (3.4) we get

$$\beta_{r\lambda}\, x^2_{\lambda}\, p_{\lambda r} + x_{\lambda}\, x_{\lambda}' \sum_{s\,(\neq r)=1}^{n} \beta_{s\lambda}'\, p\,(\lambda_r, \lambda_s') = u_{\lambda}\, p_{\lambda r} \qquad \dots (3.8)$$

If the necessary best estimator of order two exists then equations (3.5) and (3.8) must be simultaneously true. Then subtracting (3.6) from (3.8) we get

$$x_{\lambda}\, x_{\lambda}' \left[\sum_{s\,(\neq r)=1}^{n} \beta_{s\lambda}'\, p\,(\lambda_r, \lambda_s') \right] = 0 \qquad \dots (3.9)$$

Since x_λ and x_λ' are not zero and also $p(\lambda_r, \lambda_s') \neq 0$ then (3.9) implies $\beta_{s\lambda}'=0$ for all r and λ', which contradicts the conditions of unbiasedness in equation (3.2). Hence the necessary best estimator of order two does not exist in T_4-class of linear unbiased estimators.

4. The Non-Existence of Necessary Best Estimator of Order Two in T_5-Class

The general estimator for the population total in T_5-class can be defined as

$$\hat{T}_5 = \sum_{i=1}^{n} \gamma_i^{s_l} x_i \quad \ldots (4.1)$$

where $\gamma_i^{s_l}$ is the weight associated to use s_l^{th} sample whenever it includes the i^{th} unit. Thus we note that there are altogether $S' = N\binom{N-1}{n-1} = n\binom{N}{n}$ weights. The conditions of unbiasedness can be easily obtained as

$$\sum_{s_l \supset i} p(s_l)\, \gamma_i^{s_l} = 1 \text{ for all } i \quad \ldots (4.2)$$

$$\text{Var}(\hat{T}_5) = \sum_{s_l=1} p_{s_l} \Big(\sum_{i \in s_l}^{n} \gamma_i^{s_l} x_i \Big)^2 - T_x^2 \quad \ldots (4.3)$$

Minimising the R.H.S. of (4.3) subject to the condition (4.2) the optimum values of $\gamma_i^{s_l}$'s can be obtained by the equation

$$x_i\, p(s_l) \Big(\sum_{i \in s_l} \gamma_i^{s_l} x_i \Big) - \lambda_i\, p(s_l) = 0 \text{ for all } i \text{ and } s_1$$

or

$$x_i \Big[\sum_{i \in s_l} \gamma_i^{s_l} x_i - \lambda_i \Big] = 0 \text{ as } p(s_l) \neq 0 \quad \ldots. (4.4)$$

where γ_i's are the Lagrangian undetermined multipliers.

Now if we consider a population vector

$X = (x_i \neq 0, x_{i'} = 0 \text{ for } i' \neq i = 1, 2, \ldots N)$ then (4.4) gives

$$x_i^2\, \gamma_i^{s_l} = \lambda_i \quad \text{for all } i \text{ and } s_l \quad \ldots (4.5)$$

Multiplying (4.5) by $p\ (s_l)$ and then summing over those s_l's which contain i^{th} unit, we get

$$\lambda_i = \frac{x_i^2}{\sum\limits_{s_l \supset i} p_{s_l}} = \frac{x_i^2}{\pi_i} \qquad \ldots (4.6)$$

Equations (4.5) and (4.6) gives

$$\gamma_i^{s_l} = \frac{1}{\pi_i} \text{ for all } i \text{ and } s_l \qquad \ldots (4.7)$$

Equation (4.7) suggests that the necessary best estimator of order one exists in this class.

Again if we consider then population vector

$$X = (x_i \neq 0,\ x_2 \neq 0,\ x_i = 0,\ i = 3, 4, \ldots N) \qquad \ldots (4.8)$$

The possible samples from the population vector (4.8) will have the representation of the form

$$s_1 = [x_i, 0, 0 \ldots 0] \qquad \ldots (4.9)$$

$$s_2 = [0, x_2, 0 \ldots 0] \qquad \ldots (4.10)$$

$$s_3 = [x_1, x_2, 0 \ldots 0] \qquad \ldots (4.11)$$

$$s_l = [0, 0 \ldots \ldots 0] \qquad \ldots (4.12)$$

$$\text{for } l \neq 1, 2, 3,$$

Equations (4.9), (4.10), (4.11) and (4.12) alongwith the equation (4.4) give

$$x_1^2\ \gamma_1^{s_1} = \lambda_1 \qquad \ldots (4.13)$$

$$x_2^2\ \gamma_1^{s_2} = \lambda_2 \qquad \ldots (4.14)$$

$$x_1^2\ \gamma_1^{s_3} + x_1\ x_2\ \gamma_2^{s_3} = \lambda_1 \qquad \ldots (4.15)$$

$$x_1\ x_2\ \gamma_1^{s_3} + x_2^2\ \gamma_2^{s_3} = \lambda_2 \qquad \ldots (4.16)$$

The conditions of unbiasedness under (4.9), (4.10), (4.11) and (4.12) reduce to

$$p_{(s_1)}\ \gamma_1^{s_1} + p_{(s_3)}\ \gamma_1^{s_3} = 1 \qquad \ldots (4.17)$$

$$p_{(s_2)}\ \gamma_2^{s_2} + p_{(s_3)}\ \gamma_2^{s_3} = 1 \qquad \ldots (4.18)$$

The equations (4.13) to (4.18) can be solved as they involve only

6 variables. The solution is given below :

$$\lambda_1 = \frac{(x_1+x_2)\, x_1}{\left[p_{(s_1)}+p_{(s_2)}+p_{(s_3)}\right]}$$

$$\lambda_2 = \frac{(x_2+x_1)\, x_2}{\left[p_{(s_1)}+p_{(s_2)}+p_{(s_3)}\right]}$$

$$\gamma_1^{s_1} = \frac{(x_1+x_2)}{\left[p_{(s_1)}+p_{(s_2)}+p_{(s_3)}\right] x_1}$$

$$\gamma_2^{s_2} = \frac{(x_1+x_2)}{\left[p_{(s_1)}+p_{(s_2)}+p_{(s_3)}\right] x_2} \qquad \ldots(4.19)$$

$$\gamma_1^{s_3} = \frac{x_1\left(p_{(s_2)}+p_{(s_3)}\right)-p_{(s_1)}\, x_2}{\left[p_{(s_1)}+p_{(s_1)}+p_{(s_3)}\right] x_1\, p_{(s_3)}}$$

$$\gamma_2^{s_3} = \frac{x_2\left(p_{(s_1)}+p_{(s_3)}\right)-p_{(s_2)}\, x_1}{\left[p_{(s_1)}+p_{(s_2)}+p_{(s_3)}\right] x_2\, p_{(s_3)}}$$

From the set of equations in (4.19), we find that the weights depend upon the population values. Hence the necessary best estimator of order two independent of population values does not exist in T_5-class of linear unbiased estimator.

5. Non-Existence of Necessary Best Estimator of Order Two in T_6-Class of Linear Estimators

The general estimator for the population total can be defined as

$$\hat{T}_6 = \sum_{r=1}^{n} X_r\, \beta_r^{s_l} \qquad \ldots (5.1)$$

where $\beta_r^{s_l}$ is the weight to be attached to the r^{th} draw of s_l^{th} sample $\left[r=1, 2, \ldots n,\ l=1, 2, \ldots \binom{N}{n}\right]$. Thus the total number of weights in this class are $\binom{N}{n} n$.

The conditions of unbiasedness in this case come out to be

$$\sum_{r=1}^{n} \sum_{s_l \supset i_r} \beta_r^{s_l}\, p_{s_l} = 1 \qquad \ldots (5.2)$$

The second summation includes those samples in which i^{th} unit occurs at the r^{th} draw.

Also we have

$$E(\hat{T}^2_6)=\sum_{i=1}^{N} x_i^2 \sum_{r=1}^{n} \sum_{s_l \supset i_r} \beta_r^{2\, s_l} p_{(s_l)}$$

$$+\sum_{i \neq j}^{N} x_i x_j \sum_{r \neq s}^{n} \sum_{s_l \supset i_r, j_s} \beta_r^{s_l} \beta_s^{s_l} p_{(s_l^*)} \quad \ldots (5.3)$$

where $p_{(s_l^*)}$ is the probability of the sample which includes the i^{th} unit at r^{th} draw and j^{th} unit at the s^{th} draw. The variance of $(\hat{T}_6)$ is given by

$$V(\hat{T}_6)=E(\hat{T}_6)^2-T_x^2 \quad \ldots (5.4)$$

Minimising the R.H.S. of (5.4) subject to the condition (5.2), the optimum values of the weights can be obtained by the equation

$$2\sum_{i=1}^{N} x_i^2 \beta_r^{s_l} p_{(s_l)}+\sum_{i \neq j=1}^{N} x_i x_j \sum_{s(\neq r)=1}^{n} \beta_s^{s_l} p_{(s_l)}^{*}$$

$$-2\sum_{i=1}^{N} \lambda_i p_{(s_l)}=0 \quad \ldots (5.5)$$

for all s_l and r, where λ_i's are the Lagrangian undetermined multipliers.

Now let us consider the population vector

$$X=[x_1 \neq 0,\ x_j'=0 \text{ for } j\,(\neq 1)=2, 3, \ldots N].$$

Then equation (5.5) gives

$$x_1^2 \beta_r^{s_l} p_{(s_l)}-\sum_{i=1}^{N} \lambda_i p_{(s_l)}=0 \quad \ldots (5.6)$$

Summing (5.6) over those s_l's in which the i^{th} unit occurs at the r^{th} draw and then over r we get,

$$\sum_{i=1}^{N} \lambda_i=\frac{x_1^2}{\sum_{r=1}^{n} \sum_{s_l \supset i_r} p_{(s_l)}}=\frac{x_1^2}{\pi_1} \quad \ldots (5.7)$$

Equations (5.7) and (5.6) give

$$\beta_r^{s_l} = \frac{1}{\pi_1} \text{ for all } r \text{ and } s_l \qquad \dots (5.8)$$

Thus the equation (5.8) suggests that the necessary best estimator of order one exists in this class.

Let us now consider the population vector $X=(x_1 \neq 0, x_2 \neq 0, x_{j'} \neq 0$ for $j' \neq 1, 2, 3, 4, \dots N)$. Then possible samples of size n, from this vector will also have the same type of representation as we have in section 4. Of course the difference will be of order. Since the equation (5.5) is true for all samples and r; the representations of the type (4.9), (4.10) and (4.11) will give us

$$x_1^2 \beta_r^{s_l} p_{(s_l)} = \sum \lambda_i p_{(s_l)} \qquad \dots (5.9)$$

$$x_2^2 \beta_r^{s_l} p_{(s_l)} = \sum \lambda_i p_{(s_l)} \qquad \dots (5.10)$$

and

$$\left(x_1^2 + x_2^2 \right) \beta_r^{s_l} p_{(s_l)} + x_1 x_2 \sum_{s (\neq r)=1}^{n} \beta_s^{s_l} p^*_{(s_l)} = \sum \lambda_i p_{(s_l)} \qquad \dots (5.11)$$

If the second order necessary best estimator exists, it must exist for all types of populations, so that the equations (5.9), (5.10) and (5.11) must be simultaneously true.

From (5.9) and (5.10) we get

$$\beta_r^{s_l} = \frac{\Sigma \lambda_i}{2\left(x_1^2 + x_2^2 \right)} = u \text{ say} \qquad \dots (5.12)$$

Substituting the value of $\beta_r^{s_l}$ from (5.12) in (5.1) we get,

$$u \left[\left(x_1^2 + x_2^2 \right) p_{(s_l)} + x_1 x_2 \sum_{s(\neq r)=1}^{n} p^*_{(s_l)} - \frac{\left(x_1^2 + x_2^2 \right)}{2} p_{(s_l)} \right] = 0$$

or

$$u \left[\frac{\left(x_1^2 + x_2^2 \right)}{2} p_{(s_l)} + x_1 x_2 \sum_{s (\neq r)=1}^{n} p^*_{(s_l)} \right] = 0 \qquad \dots (5.13)$$

In L.H.S. of (5.13) each term within the brackets is non-zero implies $u=\beta_r^{s_l}=0$ for all r and s_l, which is a contradiction to the conditions of unbiasedness in (5.2). Thus the necessary best estimator of order two does not exist in T_6-class of linear unbiased estimators.

6. Some Results on Necessary Best Estimators

Theorem 1. If a necessary best estimator of order r, exists, then the necessary best estimators of any order less than r always exist.

Theorem 2. If a necessary best estimator of order r does not exist, then the necessary best estimator of any order higher than r does not exist.

The proofs of these theorems are based on the elementary ideas of quadratic forms and hence omitted.

Remark 1. Ajgaonkar[8] has proved that the necessary best estimators of order two exists in the most general class and is same as $\hat{Y}_{HT}$. If it is so then equation (12) p. 460 of his paper should be satisfied for all vectors of Y. Let us take the following two vectors

$$Y=[Y_1\neq 0,\ Y_j=0,\ \ j=2, 3,\ldots N]$$

and

$$Y=[Y_1\neq 0,\ Y_2\neq 0,\ Y_j=0,\ j=3, 4,\ldots N]$$

It can be easily seen that for the above vectors his equations (14) and (15) can not be obtained and which proves the fallacy in his result. Thus our result is consistent with that of Rao's[10] remark 'Prabhu Ajgaonkar's (1969) result that $\hat{Y}_{HT}$ is also the necessary best estimator of second order is incorrect'.

Finally we summarise that the necessary best estimators of order two do not exist in T_3, T_4, T_5, T_6 and T_7 classes of linear unbiased estimators. Whereas the equations (2.6), (3.7), (4.7) and (5.8) suggest that the necessary best estimators of order one exist in these classes of linear unbiased estimators. It may be noted that these results are consistent with that of Hege's[2] results.

Summary

The criterion of necessary best estimators in linear classes of unbiased estimators was introduced by Prabhu Ajgaonkar. Later he defined the necessary best estimator of various orders and pointed out that the minimum variance linear unbiased estimator (MVLUE) is a necessary best estimator of order N, the size of the finite population. In the same paper he showed that the necessary best estimator of order two exists in the most general class of unbiased linear estimators. Rao remarked that Prabhu Ajgaonkar's result that Y_{ht} is also the necessary best estimator of second order is incorrect. Hege proved the Horvitz-Thompson estimator of order one in all the

classes of linear unbiased estimators. Prabhu Ajgaonkar has established the existence of MVLUE in T_4-class under certain specified probability systems and not otherwise. But Tikkiwal (1972) in general proved the non-existence of MVLUE independent of population values in T_4-class.

In the present paper, we examined the non-existence of second order necessary best estimators, independent of population values in T_4, T_5 and T_6 classes of unbiased linear estimators. Further it has been shown that in all these classes the necessary best estimator of order one exists and it is the same as given by Horvitz and Thompson (1952).

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A CHARACTERIZATION OF THE BIVARIATE EXPONENTIAL DISTRIBUTION

BY

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Let $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ be a random sample of size $n (n \geqslant 3)$ from a bivariate population having the distribution function $F(x, y)$ where $F(x, y)=0$ if either $x \leqslant 0$ or $y \leqslant 0$ and $F(x, y)$ is continuous from the right in each argument. Without any loss of generality, we assume that

$$\min \{X_i, i=1, 2, \ldots, n\}=X_1$$

$$\min \{Y_i, i=2, 3, \ldots, n\}=Y_2$$

Then, $\min \{Y_i, i=1, 2, \ldots, n\}=\min (Y_1, Y_2)$

and $\min \{X_i, Y_i, i=1, 2, \ldots, n\}=\min (X_1, Y_1, Y_2)$

We define $\bar{F}(x, y)=P\{X>x, Y>y\}$ and the vector random variables U, V, W by

$$U=(X_3-Z, Y_3-Z, X_4-Z, Y_4-Z, \ldots, X_n-Z, Y_n-Z)$$

$$V=(X_3-X_1, X_4-X_1, \ldots, X_n-X_1)$$

$$W=(Y_3-R, Y_4-R, \ldots, Y_n-R)$$

where $R=\min (Y_1, Y_2)$

and $Z=\min (X_1, Y_1, Y_2)$

In this note we prove the following

Theorm : A necessary and sufficient set of conditions for

$\bar{F}(x, y)=e^{-\lambda_1 x-\lambda_2 y-\lambda_3 \max (x, y)}$ for some $\lambda_i>0$, $i=1, 2, 3$, is that

(*i*) U and Z are stochastically independent ;

(*ii*) V and X_1 are stochastically independent ; and

(*iii*) W and R are stochastically independent.

Proof : If $\bar{F}(x, y)=e^{-\lambda_1 x-\lambda_2 y-\lambda_3 \max(x, y)}$, $\lambda_i>0$, $i=1, 2, 3$,

then $$\bar{F}_1(x)=\bar{F}(x, 0)=e^{-(\lambda_1+\lambda_3)x}$$

and $$\bar{F}_2(y)=\bar{F}(0, y)=e^{-(\lambda_2+\lambda_3)y}.$$

It is well known [see Govindarajulu (1966)] that conditions (*ii*) and (*iii*) are both necessary and sufficient for the above requirement of exponential marginal distributions.

The joint conditional probability element of (X_3, Y_3), (X_4, Y_4), ..., (X_n, Y_n), given $Z=z$, is

$$\prod_{i=3}^{n} \frac{dF(x_i, y_i)}{\bar{F}(z, z)}, \quad 0\leqslant z\leqslant x_i, \ 0\leqslant z\leqslant y_i, \ i=3, 4, ..., n, \tag{1}$$

and zero elsewhere. Hence, the conditional probability element of U, given $Z=z$, is

$$\prod_{i=3}^{n} \frac{dF(u_{1i}+z, u_{2i}+z)}{\bar{F}(z, z)}, \quad u_{ji}\geqslant 0, \ j=1, 2; \ i=3, 4, ..., n$$

and zero elsewhere. Now, if $\bar{F}(x, y)=e^{-\lambda_1 x-\lambda_2 y-\lambda_3 \max(x, y)}$,

then $P\{U_{1i}\leqslant u_{1i}, U_{2i}\leqslant u_{2i}, i=3, 4, ..., n/Z=z\}$

$$=\prod_{i=3}^{n} \frac{\{F(u_{1i}+z, u_{2i}+z)+F(z, z)-F(z, u_{2i}+z)-F(u_{1i}+z, z)\}}{\bar{F}(z, z)}$$

does not depend on z.

We now assume that (1) does not depend on z. This implies that

$$\frac{dF(u_{13}+z, u_{23}+z)}{\bar{F}(z, z)}$$

is free of z. Hence,

$$\int_{u_{13}=s_1}^{\infty} \int_{u_{23}=s_2}^{\infty} \frac{dF(u_{13}+z, u_{23}+z)}{\bar{F}(z, z)}=\frac{\bar{F}(s_1+z, s_2+z)}{\bar{F}(z, z)}$$

does not depend on z. Allowing z to tend to 0^+, we now get

$$\bar{F}(s_1+z, s_2+z)=\bar{F}(z, z)\,\bar{F}(s_1, s_2)$$

far all $s_1 \geqslant 0$, $s_2 \geqslant 0$, $z \geqslant 0$. By Lemma 2.2 of Marshall and Olkin (1967), the solution of the functional equation in (2) is given by

$$\bar{F}(x, y)=e^{-\lambda_1 x-\lambda_2 y-\lambda_3 \max(x, y)}$$

for some $\lambda_i>0$, $i=1, 2, 3$. This completes the proof of the theorem.

Corollary : A necessary and sufficient set of conditions for

$$\bar{F}(x, y)=e^{-\lambda_1 x-\lambda_2 y-\lambda_3 \max(x, y)} \text{ for some } \lambda_i>0, i=1, 2, 3,$$

is that

(*i*) $\left(\sum_{i=3}^{n}(X_i-Z), \sum_{i=3}^{n}(Y_i-Z)\right)$ and Z are stochastically independent;

(*ii*) $\sum_{i=3}^{n}(X_i-X_1)$ and X_1 are stochastically independent; and

(*iii*) $\sum_{i=3}^{n}(Y_i-R)$ and R are stochastically independent.

It may be mentioned that the trivariate exponential distribution [see Marshall and Olkin (1967)] may be characterized in a similar way.

SUMMARY

In this note we characterize the bivariate exponential distribution using the properties of a random sample of size n ($n \geqslant 3$).

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